Department of Mathematics & Philosophy of Engineering

Faculty of Engineering Technology

The Open University of Sri Lanka

Nawala - Nugegoda

Course: MPZ 3231-Engineering Mathematics IA **Model Answer No·01**

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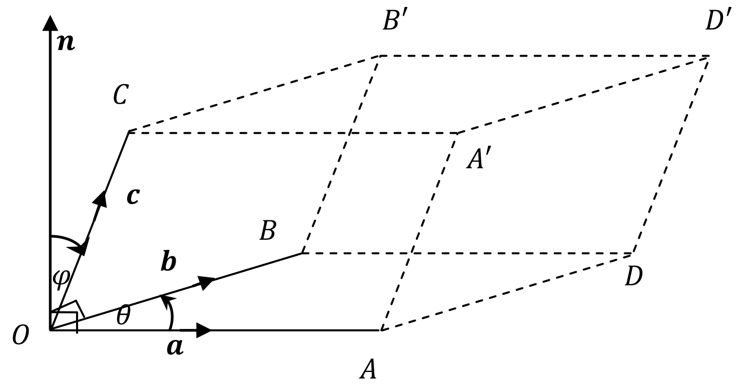
1) a) let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be three vectors, then there scalar product is written as $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}$

i) $a \times b = |a||b| \sin \theta$

Where $\boldsymbol{n}$ is the unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\boldsymbol{\theta}$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$



$\boldsymbol{a} \times \boldsymbol{b} =$ Area of the parallelogram $OADB \ \cdot \boldsymbol{n}$

$|\boldsymbol{a} \times \boldsymbol{b}| =$ Area of the parallelogram $OADB$

$\boldsymbol{a} \times \boldsymbol{b} \ \cdot \boldsymbol{c} = |\boldsymbol{a} \times \boldsymbol{b}| \cdot |\boldsymbol{c}| \cos \varphi$

Where φ is angle between $\boldsymbol{n}$ and $\boldsymbol{c}$

$|\boldsymbol{c}| \cos \varphi =$ perpendicular distance between the faces $OADB$ and $CA'D'B'$

$\boldsymbol{a} \times \boldsymbol{b} \ \cdot \boldsymbol{c} =$ Area of the parallelogram $OADB$

$\times$ perpendicular distance between the faces $OADB$ and $CA'D'B'$

= volume of the parallelepiped having $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as its co terminus edges·

ii) $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}, \boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}, \ \boldsymbol{c} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

$\therefore$ volume of the given parallelepiped = $\boldsymbol{a} \times \boldsymbol{b} \ \cdot \boldsymbol{c}$

$= [(2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}) \ (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})] \cdot (3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$$= \begin{vmatrix} i & j & k \\ 2 & -3 & 4 \\ 1 & 2 & -1 \end{vmatrix} \cdot (3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$$

$$= (-5\boldsymbol{i} + 6\boldsymbol{j} + 7\boldsymbol{k}) \cdot (3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$$

$= |-15 - 6 + 14| = 7$ units

iii) $\boldsymbol{a}' \times \boldsymbol{b}' \cdot \boldsymbol{c}'$

$$= [(\boldsymbol{a} - 2\boldsymbol{b} + 3\boldsymbol{c}) \times (-2\boldsymbol{a} + 3\boldsymbol{b} - 4\boldsymbol{c})] \cdot (\boldsymbol{a} - 3\boldsymbol{b} + 5\boldsymbol{c})$$

$$= [(-2\boldsymbol{a} \times \boldsymbol{a} + 3\boldsymbol{a} \times \boldsymbol{b} - 4\boldsymbol{a} \times \boldsymbol{c}) \quad + (\mathbf{4}\boldsymbol{b} \times \boldsymbol{a} - \mathbf{6}\boldsymbol{b} \times \boldsymbol{b} + \mathbf{8}\boldsymbol{b} \times \boldsymbol{c}) + (-6\boldsymbol{c} \times \boldsymbol{a} + 9\boldsymbol{c} \times \boldsymbol{b} - 12\boldsymbol{c} \times \boldsymbol{c})] \cdot (\boldsymbol{a} - 3\boldsymbol{b} + 5\boldsymbol{c})$$

$$= -(\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c} + 2\boldsymbol{c} \times \boldsymbol{a}) \cdot (\boldsymbol{a} - 3\boldsymbol{b} + 5\boldsymbol{c})$$

$$= -(\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{c} \cdot \boldsymbol{a} + 2\boldsymbol{c} \times \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \times \boldsymbol{b} \cdot 3\boldsymbol{b} - \boldsymbol{b} \times \boldsymbol{c} \cdot 3\boldsymbol{b} - 2\boldsymbol{c} \times \boldsymbol{a} \cdot 3\boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b} \cdot 5\boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c} \cdot 5\boldsymbol{c} + 2\boldsymbol{c} \times \boldsymbol{a} \cdot 5\boldsymbol{c})$$

$$= -\boldsymbol{b} \times \boldsymbol{c} \cdot \boldsymbol{a} + 6\boldsymbol{c} \times \boldsymbol{a} \cdot \boldsymbol{b} + 5\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}$$

$$= 0 \qquad \because [\boldsymbol{b} \times \boldsymbol{c} \cdot \boldsymbol{a} = \boldsymbol{c} \times \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}]$$

If $\boldsymbol{a}' \times \boldsymbol{b}' \cdot \boldsymbol{c}' = 0$ the volume of parallelepiped having $\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}'$ as its coterminous edges is zero·

$\therefore$ $\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}'$ are coplanar vectors·

b)

i) let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}, \boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ and $\boldsymbol{c} = c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}$

$$\boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\boldsymbol{i} + (b_3c_1 - b_1c_3)\boldsymbol{j} + (b_1c_2 - b_2c_1)\boldsymbol{k}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ (b_2c_3 - b_3c_2) & (b_3c_1 - b_1c_3) & (b_1c_2 - b_2c_1) \end{vmatrix}$$

$$= [a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)]\boldsymbol{i}$$

$$+[a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)]\boldsymbol{j}$$

$$+[a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)]\boldsymbol{k}$$

$\boldsymbol{a} \cdot \boldsymbol{c} = (a_1c_1 + a_2c_2 + a_3c_3)$

$\boldsymbol{a} \cdot \boldsymbol{b} = (a_1b_1 + a_2b_2 + a_3b_3)$

$$(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} = (a_1c_1 + a_2c_2 + a_3c_3)(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$$
$$-(a_1b_1 + a_2b_2 + a_3b_3)(c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k})$$

$$(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} = (a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1)\boldsymbol{i}$$
$$+(a_1b_2c_1 + a_2b_2c_2 + a_3b_2c_3 - a_1b_1c_2 - a_2b_2c_2 - a_3b_3c_2)\boldsymbol{j}$$
$$+\left(a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 - a_1b_{1_{c_3}} - a_2b_2c_3 - a_3b_3c_3\right)\boldsymbol{k}$$

$$(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} = \left(a_2(b_1c_2 - b_2c_1) + a_3(b_1c_3 - b_3c_1)\right)\boldsymbol{i}$$
$$+\left(a_3(b_2c_3 - b_3c_2) + a_1(b_2c_1 - b_1c_2)\right)\boldsymbol{j}$$
$$+\left(a_1(b_3c_1 - b_1c_3) + a_2(b_3c_2 - b_2c_3)\right)\boldsymbol{k}$$

$\therefore \quad \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$

ii)

$[A]$ $\quad \boldsymbol{i} \times (\boldsymbol{a} \times \boldsymbol{i}) + \boldsymbol{j} \times (\boldsymbol{a} \times \boldsymbol{j}) + \boldsymbol{k} \times (\boldsymbol{a} \times \boldsymbol{k})$

Let $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$

$(\boldsymbol{i}.\boldsymbol{i})\boldsymbol{a} - (\boldsymbol{i}.\boldsymbol{a})\boldsymbol{i} + (\boldsymbol{j}.\boldsymbol{j})\boldsymbol{a} - (\boldsymbol{j}.\boldsymbol{a})\boldsymbol{j} + (\boldsymbol{k}.\boldsymbol{k})\boldsymbol{a} - (\boldsymbol{k}.\boldsymbol{a})\boldsymbol{k}$

$=\boldsymbol{a} - (\boldsymbol{i}.\boldsymbol{a})\boldsymbol{i} + \boldsymbol{a} - (\boldsymbol{j}.\boldsymbol{a})\boldsymbol{j} + \boldsymbol{a} - (\boldsymbol{k}.\boldsymbol{a})\boldsymbol{k}$

$= \boldsymbol{3a} - [(\boldsymbol{i}.\boldsymbol{a})\boldsymbol{i} + (\boldsymbol{j}.\boldsymbol{a})\boldsymbol{j} + (\boldsymbol{k}.\boldsymbol{a})\boldsymbol{k}]$

$\boldsymbol{i}.\boldsymbol{a} = a_x, \ \boldsymbol{j}.\boldsymbol{a} = a_y, \ \boldsymbol{k}.\boldsymbol{a} = a_z$

$= \boldsymbol{3a} - \left[a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}\right]$

$=\boldsymbol{3a} - \boldsymbol{a} = 2\boldsymbol{a}$

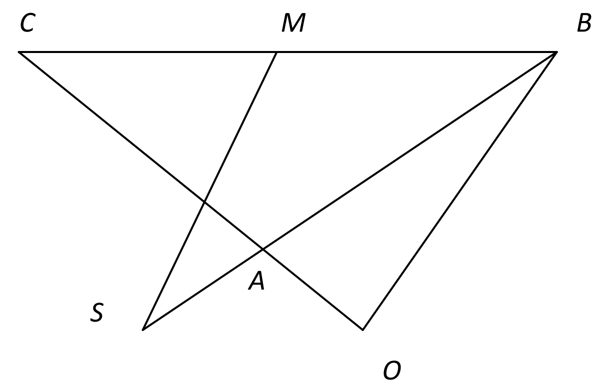
$[B]$ $\quad \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$

$\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) = (\boldsymbol{b} \cdot \boldsymbol{a})\boldsymbol{c} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$

$\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a} - (\boldsymbol{c} \cdot \boldsymbol{a})\boldsymbol{b}$

$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0}$

2) a)



i) $\overrightarrow{OA} = \boldsymbol{a},\ \overrightarrow{OC} = 3\boldsymbol{a},\ \overrightarrow{OB} = \boldsymbol{b}$

$$\overrightarrow{OM} = \frac{1}{2}\left[\overrightarrow{OB} + \overrightarrow{OC}\right]$$

$$\overrightarrow{OM} = \frac{1}{2}[\boldsymbol{b} + 3\boldsymbol{a}]$$

$$\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB} = -3\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} - 3\boldsymbol{a}$$

ii) $\overrightarrow{MB} = \frac{1}{2}\overrightarrow{CB} = \frac{1}{2}\left(\overrightarrow{CO} + \overrightarrow{OB}\right) = \frac{1}{2}(\boldsymbol{b} - 3\boldsymbol{a})$

iii) $\overrightarrow{BS} = k\overrightarrow{BA} = k(\boldsymbol{a} - \boldsymbol{b})$

$$\overrightarrow{MS} = \overrightarrow{MB} + \overrightarrow{BS} = \frac{1}{2}(\boldsymbol{b} - 3\boldsymbol{a}) + k(\boldsymbol{a} - \boldsymbol{b})$$

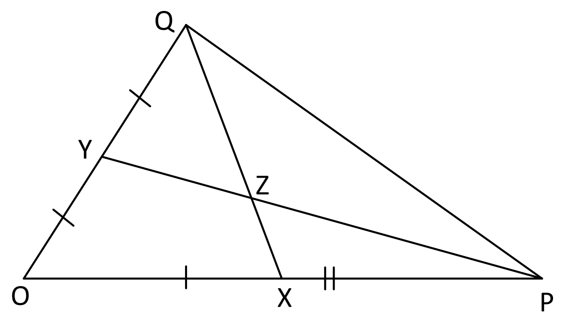
$$\overrightarrow{MS} = \overrightarrow{MB} + \overrightarrow{BS} = \left(\frac{1}{2} - k\right)\boldsymbol{b} + \left(k - \frac{3}{2}\right)\boldsymbol{a}$$

iv) If MS parallel to BO, $\overrightarrow{MS} = \mu\overrightarrow{BO} = -\mu\boldsymbol{b} \qquad \mu > 0$

$$-\mu\boldsymbol{b} = \left(\frac{1}{2} - k\right)\boldsymbol{b} + \left(k - \frac{3}{2}\right)\boldsymbol{a}$$

$$k - \frac{3}{2} = 0 \qquad \therefore\ k = \frac{3}{2}$$

b)



$\overrightarrow{OX} = \dfrac{\boldsymbol{p}}{2}$, $\overrightarrow{OY} = \dfrac{\boldsymbol{q}}{2}$

Let $XZ\colon ZQ = \lambda\colon 1$ and $YZ\colon ZP = \mu\colon 1$

$$\overrightarrow{OZ} = \frac{\lambda\boldsymbol{q} + \frac{\boldsymbol{p}}{2}}{\lambda + 1} = \frac{\mu\boldsymbol{p} + \frac{\boldsymbol{q}}{2}}{\mu + 1}$$

$$\overrightarrow{OZ} = \frac{\lambda}{\lambda + 1}\boldsymbol{q} + \frac{1}{2(\lambda + 1)}\boldsymbol{p} = \frac{\mu}{\mu + 1}\boldsymbol{p} + \frac{1}{2(\mu + 1)}\boldsymbol{q}$$

$$\therefore \frac{\lambda}{\lambda + 1} = \frac{1}{2(\mu + 1)} - - - (1)$$

and

$$\frac{1}{2(\lambda + 1)} = \frac{\mu}{\mu + 1} - - - (2)$$

$$\frac{(1)}{(2)} \quad 2\lambda = \frac{1}{2\mu}$$

$$\lambda = \frac{1}{4\mu}$$

By (2)

$$\frac{1}{2\left(\frac{1}{4\mu} + 1\right)} = \frac{\mu}{\mu + 1}$$

$$\frac{2\mu}{(4\mu + 1)} = \frac{\mu}{\mu + 1}$$

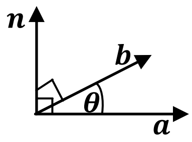
$$4\mu + 1 = 2\mu + 2$$

$$2\mu = 1$$

$$\therefore \mu = \frac{1}{2} \ \& \ \lambda = \frac{1}{2}$$

$$\overrightarrow{OZ} = \frac{\frac{1}{2}}{\frac{1}{2} + 1}\boldsymbol{q} + \frac{1}{2\left(\frac{1}{2} + 1\right)}\boldsymbol{p} = \frac{1}{3}(\boldsymbol{p} + \boldsymbol{q})$$

3) a)



$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta\, \boldsymbol{n}$

$|\boldsymbol{a}|$, $|\boldsymbol{b}|$ are modules of the vector $\boldsymbol{a}$ and $\boldsymbol{b}$, θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $\boldsymbol{n}$ is the unit vector such that if a right handed screw is rotated from $\boldsymbol{a}$ to $\boldsymbol{b}$ with the axis of the screw perpendicular to the plane contains $\boldsymbol{a}$ and $\boldsymbol{b}$

b) i) $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta$

$|\boldsymbol{a} \times \boldsymbol{b}|^{\mathbf{2}} = |\boldsymbol{a}|^{\mathbf{2}}|\boldsymbol{b}|^{\mathbf{2}} \sin^2\theta$

$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$

$(\boldsymbol{a}.\boldsymbol{b})^2 = |\boldsymbol{a}|^{\mathbf{2}}|\boldsymbol{b}|^{\mathbf{2}} \cos^2\theta$

$$\frac{|\boldsymbol{a} \times \boldsymbol{b}|^{\mathbf{2}} + (\boldsymbol{a}.\boldsymbol{b})^2}{|\boldsymbol{a}|^{\mathbf{2}}|\boldsymbol{b}|^{\mathbf{2}}} = \frac{|\boldsymbol{a}|^{\mathbf{2}}|\boldsymbol{b}|^{\mathbf{2}}(\sin^2\theta + \cos^2\theta)}{|\boldsymbol{a}|^{\mathbf{2}}|\boldsymbol{b}|^{\mathbf{2}}} = 1$$

ii) $\boldsymbol{a}.[\boldsymbol{b} \times (\boldsymbol{a} - \boldsymbol{b})] = \boldsymbol{a}.\boldsymbol{b} \times \boldsymbol{a} - \boldsymbol{a}.\boldsymbol{b} \times \boldsymbol{b} = \mathbf{0}$

c) $\boldsymbol{p} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{q} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

If $PQRS$ is a parallelogram $\overrightarrow{PQ} = \overrightarrow{SR}$

$\overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{SO} + \overrightarrow{OR}$

$\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OR} - \overrightarrow{OQ}$

$\overrightarrow{OS} = \boldsymbol{p} + \boldsymbol{r} - \boldsymbol{q} \quad --(S)$

Given that $\overrightarrow{PQ} \times \overrightarrow{PR} = \overrightarrow{OR}$

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -(2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) + (\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = -(2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) + (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$

$= (x-2)\boldsymbol{i} + (y+3)\boldsymbol{j} + (z-1)\boldsymbol{k}$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & 1 \\ (x-2) & (y+3) & (z-1) \end{vmatrix}$$

$\overrightarrow{PQ} \times \overrightarrow{PR} = [2(z-1) - (y+3)]\boldsymbol{i} + [(x-2) + (z-1)]\boldsymbol{j}$

$+[-(y+3) - 2(x-2)]\boldsymbol{k}$

$\overrightarrow{PQ} \times \overrightarrow{PR} = \overrightarrow{OR} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

$2z - y - 5 = x \qquad x + z - 3 = y \qquad -2x - y + 1 = z$

$x + y - 2z = -5 \quad --(1)$

$x - y + z \ = 3 \quad --(2)$

$2x + y + z = 1 \quad --(3)$

$(1) + (2) \qquad 2x - z \ = -2 \quad --(4)$

$(2) + (3) \qquad 3x + 2z = 4 \quad --(5)$

$(4) \times 2 + (5) \qquad 7x = 0$

$x = 0$

$z = 2$

$y = -1$

$\overrightarrow{OR} = -\boldsymbol{j} - 2\boldsymbol{k} \qquad |\boldsymbol{r}| = \sqrt{5}$

From equation (S)

$\overrightarrow{OS} = \boldsymbol{p} + \boldsymbol{r} - \boldsymbol{q}$

$= (2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) + (-\boldsymbol{j} - 2\boldsymbol{k}) - (\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$= \boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

Area of the parallelogram $\overrightarrow{PQ} \times \overrightarrow{PR}$

$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = (x-2)\boldsymbol{i} + (y+3)\boldsymbol{j} + (z-1)\boldsymbol{k} = -2\boldsymbol{i} \pm 3\boldsymbol{j} + \boldsymbol{k}$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & 1 \\ -2 & 2 & 1 \end{vmatrix}$$

$\overrightarrow{PQ} \times \overrightarrow{PR} = 0\boldsymbol{i} - (-1+2)\boldsymbol{j} + (-2+4)\boldsymbol{k} = -\boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{PQ} \times \overrightarrow{PR} = -\boldsymbol{j} + 2\boldsymbol{k}$

$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \sqrt{5}$

4) a) If $A = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 2 & 1-\lambda & 2 \\ 2 & -1 & 4-\lambda \end{vmatrix} = 0$ for some values for λ

Then the matrix A invertible for those values of λ

$(1-\lambda)[(1-\lambda)(4-\lambda)+2] - 1[2(4-\lambda)-4] - 1[-2-2(1-\lambda)] = 0$

$(1-\lambda)[\lambda^2 - 5\lambda + 6] - (4-2\lambda) + (4-2\lambda) = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda + 6 = 0$

$(\lambda+1)(\lambda^2 - 5\lambda - 6) = 0$

$(\lambda+1)(\lambda-2)(\lambda-3) = 0$

$\lambda = -1,\ \lambda = 2$ and $\lambda = 3$

$\therefore$ Matrix A invertible for $\lambda = -1,\ \lambda = 2$ and $\lambda = 3$

b) $x + 2y + 3z = kx$

$3x + y + 2z = ky$

$2x + 3y + z = kz$

$(1-k)x + 2y + 3z = 0$

$3x + (1-k)y + 2z = 0$

$2x + 3y + (1-k)z = 0$

$$\begin{bmatrix} 1-k & 2 & 3 \\ 3 & 1-k & 2 \\ 2 & 3 & 1-k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1-k & 2 & 3 \\ 3 & 1-k & 2 \\ 2 & 3 & 1-k \end{bmatrix}$$

$|A| = (1-k)[(1-k)(1-k)-6] - 2[3(1-k)-4] + 3[9-2(1-k)]$

$|A| = (1-k)^3 - 18(1-k) + 35$

$= y^3 - 18y + 36$ where $y = 1-k$

$= y^3 - 18y + 36$

$= (y+5)(y^2 - 5y + 7)$

$|A| = 0$ When $y = -5$

$\therefore 1 - k + 5 = 0$

$k = 6$

The above systems of equation have non zero solutions when $k = 6$

c) $x + y = 2$

$x + (2-\lambda)y + z = 2$

$2\lambda y + \cos\lambda\, z = 3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 2\lambda & \cos\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 2\lambda & \cos\lambda \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$Ax = B$

Augmented Matrix $[A, B] = \begin{bmatrix} 1 & 1 & 0 & : & 2 \\ 1 & 2-\lambda & 1 & : & 2 \\ 0 & 2\lambda & \cos\lambda & : & 3 \end{bmatrix}$

$$C_1 \to C_1 - C_2 \begin{bmatrix} 0 & 1 & 0 & : & 2 \\ \lambda-1 & 2-\lambda & 1 & : & 2 \\ -2\lambda & 2\lambda & \cos\lambda & : & 3 \end{bmatrix}$$

$$C_2 \to \frac{C_4}{2} - C_2 \begin{bmatrix} 0 & 0 & 0 & : & 2 \\ \lambda-1 & \lambda-1 & 1 & : & 2 \\ -2\lambda & {}^{3}/_{2} - 2\lambda & \cos\lambda & : & 3 \end{bmatrix}$$

$\therefore$ Rank of $[A, B] = 3$

$|A| = [(2-\lambda)\cos\lambda - 2\lambda] - \cos\lambda$

$\quad = \cos\lambda\,(1-\lambda) - 2\lambda$

$\quad = -\lambda(2+\cos\lambda) + \cos\lambda$

If $-\lambda(2+\cos\lambda) + \cos\lambda = 0$

$\therefore |A| = 2$

$\therefore$ Rank of $A = 2$ where $\lambda(2+\cos\lambda) + \cos\lambda = 0$

Rank of $A \neq$ Rank of $[A, B]$

$\therefore$ Then the system has no solutions, so, that is the system is inconsistent.

When $\lambda = 0$ rank of $(A, B) = 3$

$|A| = \cos 0 = 1 \quad \therefore |A| \neq 0$

$\therefore$ Rank of $A == 3$

Rank of $A =$ Rank of $[A, B]$

$\therefore$The system has a unique solution.

If $\lambda = 0$

$Ax = B$

The augmented matrix $A, B = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$$R_2 \to R_2 - R_1 \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} R_2 \to R_2 - R_3 \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$R_1 \to R_1 - R_2 \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 3 \end{bmatrix}$$

$\therefore z = 3, y = -3, x = 5$

5. a) $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 4 \\ -4 & 1 & 0 \\ 8 & 0 & -3 \end{bmatrix}$$

$$A^3 - A - I = \begin{bmatrix} 5 & -4 & 4 \\ -4 & 1 & 0 \\ 8 & 0 & -3 \end{bmatrix} - 4\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

b) $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -(2+\lambda) \end{bmatrix}$

$$= (1-\lambda)[-(3-\lambda)(2+\lambda)]$$
$$= (\lambda-1)(3-\lambda)(2+\lambda)$$

Since $|A - \lambda I| = 0$

$\therefore (\lambda-1)(3-\lambda)(2+\lambda)$

$\therefore\ \lambda = 1,\ \lambda = 3$ and $\lambda = -2$

c) $x - 2y + 3z = 4$

$2x - 3y + az = 5$

$3x - 4y + 5z = b$

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & a \\ 3 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ b \end{bmatrix}$$

$AX = B$

$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & a \\ 3 & -4 & 5 \end{bmatrix}$ $\qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\qquad B = \begin{bmatrix} 4 \\ 5 \\ b \end{bmatrix}$

Det $A = 1(-15 + 4a) - 2(3a - 10) + 3(-8 + 9)$

$= 8 - 2a$

i) When $a = 4$ then Det $A = 0$

When $a = 4$ then Rank $A = (3 - 1) = 2$

$$A:B = \begin{bmatrix} 1 & -2 & 3 & : & 4 \\ 2 & -3 & 4 & : & 5 \\ 3 & -4 & 5 & : & b \end{bmatrix}$$

$$R_3 \to R_2 \times 2 - R_1 - R_3 \begin{bmatrix} 1 & -2 & 3 & : & 4 \\ 2 & -3 & 4 & : & 5 \\ 0 & 0 & 0 & : & 6-b \end{bmatrix}$$

$\therefore$ If $b = 6$, Det $(A:B) = 0$

Rank $(A:B) = 2$

When $a = 4$ and $b \neq 6$ then Det $(A:B) \neq 0$

Rank $(A:B) = 3$, Rank $A = 2$

$\therefore$ The system is inconsistent.

$\therefore$ That is the system has no solution

ii. When $a \neq 4$, then Det $A \neq 0$

Rank $A = 3 =$ Rank $(A:B)$

$\therefore$ The system has unique solution

ii. When $b = 6$ and $a = 4$,

Rank $(A:B) =$ Rank $A \nless$ number of unknowns

$\therefore$ The system has infinite number of solution

6. a)

$$\Delta = \begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$$

$\downarrow R_1 \times a, R_2 \times b, R_3 \times c$

$$\Delta = \frac{1}{abc} \begin{vmatrix} a & abc & abc(b+c) \\ b & abc & abc(c+a) \\ c & abc & abc(a+b) \end{vmatrix}$$

$$\Delta = \frac{(abc)^2}{abc} \begin{vmatrix} a & 1 & (b+c) \\ b & 1 & (c+a) \\ c & 1 & (a+b) \end{vmatrix} = abc \begin{vmatrix} a & 1 & (b+c) \\ b & 1 & (c+a) \\ c & 1 & (a+b) \end{vmatrix}$$

b)

$3x + y + z = px$

$x + 3y + z = py$

$x + y + 3z = pz$

$$\begin{bmatrix} 3-p & 1 & 1 \\ 1 & 3-p & 1 \\ 1 & 1 & 3-p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Ax = 0$

Where $A = \begin{bmatrix} 3-b & 1 & 1 \\ 1 & 3-b & 1 \\ 1 & 1 & 3-b \end{bmatrix}$

The homogeneous liner equation system $Ax = 0$ has non trivial solution if and only if Det $A = 0$

$\therefore (3-p)[(3-p)^2 - 1] - 1[(3-p) - 1] + 1[1 - (3-p)] = 0$

$(3-p)^3 - 3(3-p) + 2 = 0$

Let $3 - p = t$

$t^3 - 3t + 2 = 0$

$(t-1)(t^2 - t - 2) = 0$

$(t-1)(t-2)(t+1) = 0$

$t = 1,\ t = 2$ or $t = -1$

$3 - p = 1,\ 3 - p = 2$ or $3 - p = -1$

$p = 2,\ p = 1$ or $3 - p = 1$ when $p = 1, 2$ or 4 the system of equation has non trivial solution

c) $2x + y = 0$

$(k-2)x + ky - 2z = 0$

$6x + 3y + (k-1)z = 0$

$$\begin{bmatrix} 2 & 1 & -1 \\ k-2 & k & -2 \\ 6 & 3 & k-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Ax = 0$

Where $A = \begin{bmatrix} 2 & 1 & -1 \\ k-2 & k & -2 \\ 6 & 3 & k-1 \end{bmatrix}$

$\therefore$ Det $A = 2[k(k-1) + 6] - 1[(k-2)(k-1) + 12] - 1[3(k-2) - 6k]$

$= 2[k^2 - k + 6] - [k^2 - 3k + 14] - [-6 - 3k]$

$= k^2 + 4k + 4$

$= (k+2)^2$

When $k = -2$, Det $A = 0$

$\therefore$ There is only one value for k $(k = 2)$ which the given system of equation has non trivial solution